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On the pointwise convergence of Cesàro means of two-variable functions with respect to unbounded Vilenkin systems ☆

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Abstract

One of the most celebrated problems in dyadic harmonic analysis is the pointwise convergence of the Fejér (or $(C,1)$) means of functions on unbounded Vilenkin groups. There was no known positive result before the author's paper appeared in 1999 (J. Approx. Theory 101(1) (1999) 1) with respect to the a.e. convergence of the one-dimensional $(C,1)$ means of L^p $(p>1)$ functions. This paper is concerned with the almost everywhere convergence of a subsequence of the two-dimensional Fejér means of functions in $L\log^+ L$. Namely, we prove the a.e. relation $\lim_{n,k\to\infty} \sigma_{M_n,\tilde{M}_k} f = f$ (for the indices the condition $|n-k|<\alpha$ is provided, where $\alpha>0$ is some constant).

MSC: 42C10

Keywords: Two-variable integrable functions; $L\log^+L$ space; Unbounded Vilenkin groups; Vilenkin series; $(C,1)$ means; a.e. convergence

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1. Introduction

One of the most celebrated problems in dyadic harmonic analysis is the pointwise convergence of the Fejér (or $(C,1)$) means of functions on one- and two-dimensional unbounded Vilenkin groups.

Fine [6] proved every Walsh–Fourier series (in the Walsh case $m_j = 2$ for all $j \in \mathbb{N}$) is a.e. (C,α) summable for $\alpha > 0$. His argument is an adaptation of the older trigonometric analogue due to Marcinkiewicz [17]. Schipp [21] gave a simpler proof for the case $\alpha = 1$, i.e. $\sigma_n f \to f$ a.e. ($f \in L^1(G_m)$). He proved that σ^* is of weak type (L^1, L^1). That σ^* is bounded from H^1 to L^1 was discovered by Fujii [7]. (See also [23].)

The theorem of Schipp are generalized to the p-series fields ($m_j = p$ for all $j \in \mathbb{N}$) by Taibleson [24], and later to bounded Vilenkin systems by Pál and Simon [19].

The methods known in the trigonometric or in the Walsh, bounded Vilenkin case are not powerful enough. One of the main problems is that the proofs on the bounded Vilenkin groups (or in the trigonometric case) heavily use the fact that the L^1 norm of the Fejér kernels are uniformly bounded. This is not the case if the group G_m is an unbounded one [20]. From this it follows that the original theorem of Fejér does not hold on unbounded Vilenkin groups. Namely, Price proved [20] that for an arbitrary sequence m ($\sup_n m_n = \infty$) and $a \in G_m$ there exists a function f continuous on G_m and $\sigma_n f(a)$ does not converge to $f(a)$. Moreover, he proved [20] that if $\frac{\log m_n}{M_n} \to \infty$, then there exists a function f continuous on G_m whose Fourier series are not $(C,1)$ summable on a set $S \subset G_m$ which is non-denumerable. That is, only, a.e. convergence can be stated for unbounded Vilenkin groups. The almost everywhere convergence of the full partial sums for L^p, $p > 1$, is known in the bounded case [15] but not in the unbounded case. On the other hand, mean convergence of the full partial sums for L^p, $p > 1$, is known for the unbounded case. Namely, in 1999 the author [11] proved that if $f \in L^p(G_m)$, where $p > 1$, then $\sigma_n f \to f$ almost everywhere. This was the very first "positive" result with respect to the a.e. convergence of the Fejér means of functions on unbounded Vilenkin groups. Later, the author of this paper gave a partial answer for L^1 case. He discussed a partial sequence of the sequence of the Fejér means. Namely, if $f \in L^1(G_m)$, then he proved (see [13]) that $\sigma_{M_n} f \to f$ almost everywhere.

What can be said in the case of two-dimensional functions? This is "another story". For double trigonometric Fourier series, Marcinkiewicz and Zygmund [16] proved that $\sigma_{m,n} f \to f$ a.e. as $m, n \to \infty$ provided the integral lattice points (m,n) remain in some positive cone, that is provided $\beta^{-1} \leqslant m/n \leqslant \beta$ for some fixed parameter $\beta \geqslant 1$. It is known that the classical Fejér means are dominated by decreasing functions whose integrals are bounded but this fails to hold for the one-dimensional Walsh–Fejér kernels. This growth difference is exacerbated in higher dimensions so that the trigonometric techniques are not powerful enough for the Walsh case.

In 1992 Móricz et al. [18] proved that $\sigma_{2^{n_1},2^{n_2}} f \to f$ a.e. for each $f \in L^1([0,1)^2)$, when $n_1, n_2 \to \infty$, $|n_1 - n_2| \leqslant \alpha$ for some fixed α. Later, Gát and Weisz proved (independently, in the same year) this for the whole sequence, that is, the theorem of Marcinkiewicz and Zygmund with respect to the Walsh–Paley system (see [9,26]). In 2000 Blahota and the author of this paper generalized this theorem with respect to two-dimensional bounded Vilenkin systems [2].

If we do not provide a "cone restriction" for the indices in $\sigma_{n,k} f$ that is, we discuss the convergence of this two-dimensional Fejér means in the Pringsheim sense, then the situation changes. In 1992 Móricz et al. [18] proved with respect to the Walsh–Paley system that $\sigma_{n,k} f \to f$ a.e. for each $f \in L \log^+ L([0,1)^2)$, when $\min\{n,k\} \to \infty$. Later, in 2002 Weisz generalized [27] this with respect to two-dimensional bounded Vilenkin systems. In 2000 Gát proved [12] that the theorem of Móricz et al. above cannot be improved. Namely, let $\delta : [0,+\infty) \to [0,+\infty)$ be a measurable function with property $\lim_{t\to\infty} \delta(t) = 0$. Gát proved the existence of a function $f \in L^1([0,1)^2)$ such that $f \in L \log^+ L\delta(L)$, and $\sigma_{n,k} f$ does not converge to f a.e. as $\min\{n,k\} \to \infty$.

What can be said in the two-dimensional case with respect to unbounded Vilenkin systems? Nothing before this paper. We prove the following theorem. Let $f \in (L \log^+ L)(G_m \times G_{\tilde{m}})$. Then we have $\sigma_{M_{n_1},\tilde{M}_{n_2}} f \to f$ almost everywhere, where $\min\{n_1, n_2\} \to \infty$ provided that the distance of the indices is bounded, that is, $|n_1 - n_2| < \alpha$ for some fixed constant $\alpha > 0$.

It seems also to be interesting to discuss the almost everywhere convergence of Marcinkiewicz means $\frac{1}{n}\sum_{j=0}^{n-1} S_{j,j} f$ of integrable functions on unbounded groups. Although, this mean is defined for two-variable functions, in the view of almost everywhere convergence there are similarities with the one-dimensional case. For the trigonometric, Walsh–Paley, and bounded Vilenkin case see the papers of Zhizhiasvili, Goginava, and Gát [28,14,8].

Next, we give a brief introduction to the theory of Vilenkin systems. These orthonormal systems were introduced by Vilenkin in 1947 (see e.g. [25,1]) as follows.

Let $m := (m_k, k \in \mathbb{N})$ ($\mathbb{N} := \{0,1,\dots\}, \mathbb{P} := \mathbb{N}\setminus\{0\}$) be a sequence of integers each of them not less than 2. Let Z_{m_k} denote the discrete cyclic group of order m_k. That is, Z_{m_k} can be represented by the set $\{0,1,\dots,m_k-1\}$, with the group operation $\bmod m_k$ addition. Since the groups is discrete, then every subset is open. The normalized Haar measure on Z_{m_k}, μ_k is defined by $\mu_k(\{j\}) := 1/m_k$ ($j \in \{0,1,\dots,m_k-1\}$). Let

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}.$$

Then every $x \in G_m$ can be represented by a sequence $x = (x_i, i \in \mathbb{N})$, where $x_i \in Z_{m_i}$ ($i \in \mathbb{N}$). The group operation on G_m (denoted by $+$) is the coordinate-wise addition (the inverse operation is denoted by $-$), the measure (denoted by μ), which is the normalized Haar measure, and the topology are the product measure and topology. Consequently, G_m is a compact Abelian group. If $\sup_{n\in\mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. The Vilenkin group is metrizable in the following way:

$$d(x,y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{M_{i+1}} \quad (x,y \in G_m).$$

The topology induced by this metric, the product topology, and the topology given by intervals defined below, are the same. A base for the neighborhoods of G_m can be given by the intervals:

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \text{ for } i < n\}$$

for $x \in G_m, n \in \mathbb{P}$. Let $0 = (0, i \in \mathbb{N}) \in G_m$ denote the nullelement of G_m.

Furthermore, let $L^p(G_m)$ $(1 \leqslant p \leqslant \infty)$ denote the usual Lebesgue spaces ($||.||_p$ the corresponding norms) on G_m, $\mathscr{A}_n$ the σ-algebra generated by the sets $I_n(x)$ $(x \in G_m)$, and E_n the conditional expectation operator with respect to $\mathscr{A}_n$ $(n \in \mathbb{N})$ $(f \in L^1)$.

The concept of the maximal Hardy space [22] $H^1(G_m)$ is defined by the maximal function $f^* := \sup_n |E_n f|$ $(f \in L^1(G_m))$, saying that f belongs to the Hardy space $H^1(G_m)$ if $f^* \in L^1(G_m)$. $H^1(G_m)$ is a Banach space with the norm $||f||_{H^1} := ||f^*||_1$. We say that the function $f \in L^1(G_m)$ belongs to the logarithm space $L\log^+ L(G_m)$ if the integral

$$||f||_{L\log^+ L} := \int_{G_m} |f(x)| \log^+(|f(x)|)\, d\mu(x)$$

is finite. The positive logarithm $\log^+$ is defined as

$$\log^+(x) := \begin{cases} \log(x) & \text{if } x > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let X and Y be either $H^1(G_m)$ or $L^p(G_m)$ for some $1 \leqslant p \leqslant \infty$ with norms $||.||_X$ and $||.||_Y$. We say that operator T is of type (X, Y) if there exist an absolute constant $C > 0$ for which $||Tf||_Y \leqslant C||f||_X$ for all $f \in X$. If $X = Y = L^p(G_m)$ then we often say that T is of type (p,p) instead of type (L^p, L^p). T is of weak type (L^1, L^1) (or weak type $(1,1)$) if there exist an absolute constant $C > 0$ for which $\mu(Tf > \lambda) \leqslant C||f||_1/\lambda$ for all $\lambda > 0$ and $f \in L^1(G_m)$. It is known that the operator which maps a function f to the maximal function f^* is of weak type (L^1, L^1), and of type (L^p, L^p) for all $1 < p \leqslant \infty$ (see e.g. [3]).

Let $M_0 := 1, M_{n+1} := m_n M_n$ $(n \in \mathbb{N})$ be the so-called generalized powers. Then each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0, 1, \ldots, m_i - 1\},\ i \in \mathbb{N}),$$

where only a finite number of n_i's differ from zero. The generalized Rademacher functions are defined as

$$r_n(x) := \exp\left(2\pi\imath \frac{x_n}{m_n}\right) \quad (x \in G_m, n \in \mathbb{N},\ \imath := \sqrt{-1}).$$

It is known that $\sum_{i=0}^{m_n-1} r_n^i(x) = \begin{cases} 0 & \text{if } x_n \neq 0 \\ m_n & \text{if } x_n = 0 \end{cases}$ $(x \in G_m,\ n \in \mathbb{N})$. The nth Vilenkin function is

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} \quad (n \in \mathbb{N}).$$

The system $\psi := (\psi_n : n \in \mathbb{N})$ is called a Vilenkin system. Each ψ_n is a character of G_m, and all the characters of G_m are of this form. Define the m-adic addition as

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j \,(\operatorname{mod} m_j))M_j \quad (k, n \in \mathbb{N}).$$

Then, $\psi_{k \oplus n} = \psi_k \psi_n,\ \psi_n(x+y) = \psi_n(x)\psi_n(y),\ \psi_n(-x) = \bar{\psi}_n(x),\ |\psi_n| = 1\ (k, n \in \mathbb{N}, x, y \in G_m)$.

Define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the Fejér means, and the Fejér kernels with respect to the Vilenkin system ψ as follows:

$$\hat{f}(n) := \int_{G_m} f \bar{\psi}_n,$$

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k,$$

$$D_n(y, x) = D_n(y - x) := \sum_{k=0}^{n-1} \psi_k(y)\bar{\psi}_k(x),$$

$$\sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_k f,$$

$$K_n(y, x) = K_n(y - x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k(y - x),$$

$$\left(n \in \mathbb{P}, y, x \in G_m,\ \hat{f}(0) := \int_{G_m} f, \quad S_0 f = D_0 = K_0 = 0, \quad f \in L^1(G_m)\right).$$

It is well known that

$$S_n f(y) = \int_{G_m} f(x) D_n(y - x)\, d\mu(x),$$

$$\sigma_n f(y) = \int_{G_m} f(x) K_n(y - x)\, d\mu(x) \quad (n \in \mathbb{P},\ y \in G_m,\ f \in L^1(G_m)).$$

It is also well known that

$$D_{M_n}(x) = \begin{cases} M_n & \text{if } x \in I_n(0), \\ 0 & \text{if } x \notin I_n(0), \end{cases}$$

$$S_{M_n} f(x) = M_n \int_{I_n(x)} f = E_n f(x) \quad (f \in L^1(G_m), n \in \mathbb{N}).$$

Let $\tilde{m}$ be a sequence like m. The relation between the sequence $(\tilde{m}_n)$ and $(\tilde{M}_n)$ is the same as between sequence (m_n) and (M_n). The group $G_m \times G_{\tilde{m}}$ is called a two-dimensional Vilenkin group. The normalized Haar measure is denoted by μ, just as in the one-dimensional case. It will not cause any misunderstood. In my opinion, it is always unambiguous the dimension of the set and measure we are talking about.

The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, the Dirichlet kernels, the Fejér means, and the Fejér kernels with respect to the two-dimensional Vilenkin system are defined as

$$\hat{f}(n_1, n_2) := \int_{G_m \times G_{\tilde{m}}} f(x^1, x^2) \bar{\psi}_{n_1}(x^1) \bar{\psi}_{n_2}(x^2) \, d\mu(x^1, x^2),$$

$$S_{n_1,n_2} f(y^1, y^2) := \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \hat{f}(k_1, k_2) \psi_{k_1}(y^1) \psi_{k_2}(y^2),$$

$$\begin{aligned} D_{n_1,n_2}(y, x) = D_{n_1,n_2}(y - x) &= D_{n_1}(y^1 - x^1) D_{n_2}(y^2 - x^2) \\ &:= \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \psi_{k_1}(y^1) \psi_{k_2}(y^2) \bar{\psi}_{k_1}(x^1) \bar{\psi}_{k_2}(x^2), \end{aligned}$$

$$\sigma_{n_1,n_2} f(y^1, y^2) := \frac{1}{n_1 n_2} \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} S_{k_1,k_2} f(y^1, y^2),$$

$$K_{n_1,n_2}(y, x) = K_{n_1,n_2}(y - x) := \frac{1}{n_1 n_2} \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} D_{k_1,k_2}(y - x),$$

$$(y = (y^1, y^2), x = (x^1, x^2) \in G_m \times G_{\tilde{m}}).$$

It is also well known that

$$\sigma_{n_1,n_2} f(y) = \int_{G_m \times G_{\tilde{m}}} f(x) K_{n_1,n_2}(y - x) \, d\mu(x),$$

$$S_{M_{n_1},\tilde{M}_{n_2}}f(x) = M_{n_1}\tilde{M}_{n_2}\int_{I_{n_1}(x^1)\times I_{n_2}(x^2)} f = (E^1_{n_1}\otimes E^2_{n_2})f(x).$$

2. The theorem

The aim of this paper is to give a partial answer for the two-dimensional case. We discuss a partial sequence of the sequence of the Fejér means. Namely, we prove:

Theorem 2.1. *Let $f\in(L\log^+ L)(G_m\times G_{\tilde{m}})$. Then we have $\sigma_{M_{n_1},\tilde{M}_{n_2}}f\to f$ almost everywhere, where $\min\{n_1,n_2\}\to\infty$ provided that the distance of the indices is bounded, that is, $|n_1-n_2|<\alpha$ for some fixed constant $\alpha>0$.*

In order to prove Theorem 2.1 we need several lemmas. The first one is the so-called Calderon–Zygmund decomposition Lemma [5] on unbounded Vilenkin groups in the one-dimensional case (for the proof see e.g. [23,10]). For $z\in G_m,\ k\in\mathbb{N},\ j\in\{0,\dots,m_k-1\}$ we use the notation

$$I_k(z,j) = I_{k+1}(z_0,\dots,z_{k-1},j) = \{x\in I_k(z): x_k = j\}.$$

Lemma 2.2. *Let $f\in L^1(G_m)$, and $\lambda>\|f\|_1>0$ arbitrary. Then the function f can be decomposed in the following form:*

$$f = f_0 + \sum_{j=1}^{\infty} f_j,\quad \|f_0\|_\infty\leqslant C\lambda,\quad \|f_0\|_1\leqslant C\|f\|_1,$$

$$\operatorname{supp} f_j\subset\bigcup_{l=\alpha_j}^{\beta_j} I_{k_j}(z^j,l) = J_j,\quad \int_{G_m} f_j\,d\mu = 0\quad (j\in\mathbb{P}),$$

and for

$$F = \bigcup_{j\in\mathbb{P}} J_j,\quad \mu(F)\leqslant C\frac{\|f\|_1}{\lambda}.$$

Moreover, the sets J_j are disjoint $(j\in\mathbb{P})$.

The second one is as follows. For an integrable one-variable function f, and $1\leqslant A\in\mathbb{N}$ we define the operators $L_{1,A}, H_{1,A}$ in the following way:

$$L_{1,A}f(y) := M_{A-1}\int_{I_{A-1}(y)\setminus I_A(y)} f(x)\frac{1}{1-r_{A-1}(y-x)}\,d\mu(x),$$

$$H_{1,A}f := |L_{1,A}f|$$

$(f \in L^1(G_m), y \in G_m)$. The maximal operator H_1 is defined as

$$H_1 f := \sup_{1 \leqslant A \in \mathbb{N}} H_{1,A} f \quad (f \in L^1(G_m)).$$

In 2003 the author proved [13] that the operator H_1 is of type (L^2, L^2) with respect to any (bounded or not) Vilenkin group.

Let $f \in L^1(G_m)$ such that

$$\int_{G_m} f \, d\mu = 0, \quad \operatorname{supp} f \subset \bigcup_{j=\alpha}^{\beta} I_k(z,j) =: I,$$

where $I_k(z,j) = I_{k+1}(z_0, \dots, z_{k-1}, j)$, $z \in G_m$, and $j \in \{\alpha, \alpha+1, \dots, \beta\} \subset \{0, 1, \dots, m_k - 1\}$. Let $\gamma := \lfloor (\alpha+\beta)/2 \rfloor$. Define the distance of $j, k \in \{0, 1, \dots, m_k - 1\} = Z_{m_k}$ as

$$\rho(j,k) := \begin{cases} |j-k| & \text{if } |j-k| \leqslant \dfrac{m_k}{2}, \\ m_k - |j-k| & \text{if } |j-k| > \dfrac{m_k}{2}. \end{cases}$$

In other words, Z_{m_k} is considered as a circle. Define the set $6I$ in the following way:

If $\beta - \alpha + 1 \geqslant m_k/12$, then $6[\alpha, \beta] := \{0, \dots, m_k - 1\}$,

$$6I := \bigcup_{j \in 6[\alpha,\beta]} I_k(z,j) = I_k(z).$$

On the other hand, if $\beta - \alpha + 1 < m_k/12$, then $6[\alpha,\beta] := \{j \in Z_{m_k} : \rho(j,\gamma) \leqslant 3(\beta - \alpha + 1)\}$,

$$6I := \bigcup_{j \in 6[\alpha,\beta]} I_k(z,j).$$

It is obvious that $\mu(I) \leqslant \mu(6I) \leqslant 12\mu(I)$. Denote by $e_k \in G_m$ the sequence whose kth coordinate is 1, and the rest are zeros $(k \in \mathbb{N})$.

Next, we prove for the operator H_1:

Lemma 2.3.

$$\int_{G_m \setminus 6I} H_1 f \, d\mu \leqslant C \|f\|_1.$$

Proof. For $x \in I$, and $y \in \bigcup_{j \in Z_{m_k} \setminus 6[\alpha,\beta]} I_k(z,j)$ we give an upper bound for

$$\left| \frac{1}{1 - r_k(y-x)} - \frac{1}{1 - r_k(y - \gamma e_k)} \right|$$

and later for the sum of them. The definition of ρ gives

$$\frac{1}{\left| \sin\left(\pi \frac{y_k - x_k}{m_k}\right) \right|} = \frac{1}{\sin\left(\pi \frac{\rho(y_k, x_k)}{m_k}\right)} \leqslant C \frac{m_k}{\rho(y_k, x_k)}.$$

Since

$$\frac{1}{1-\exp(2\pi\imath z)}=\frac{1}{2}+\frac{\imath}{2}\cot(\pi z),$$

then we have

$$\begin{aligned}
&\left|\frac{1}{1-r_k(y-x)}-\frac{1}{1-r_k(y-\gamma e_k)}\right| \\
&\quad=\frac{1}{2}\left|\frac{\cos\left(\pi\frac{y_k-x_k}{m_k}\right)}{\sin\left(\pi\frac{y_k-x_k}{m_k}\right)}-\frac{\cos\left(\pi\frac{y_k-\gamma}{m_k}\right)}{\sin\left(\pi\frac{y_k-\gamma}{m_k}\right)}\right| \\
&\quad=\frac{1}{2}\left|\frac{\sin\left(\pi\frac{x_k-\gamma}{m_k}\right)}{\sin\left(\pi\frac{y_k-x_k}{m_k}\right)\sin\left(\pi\frac{y_k-\gamma}{m_k}\right)}\right| \\
&\quad\leqslant C\frac{(\beta-\alpha+1)/m_k}{\left|\sin\left(\pi\frac{y_k-x_k}{m_k}\right)\sin\left(\pi\frac{y_k-\gamma}{m_k}\right)\right|} \\
&\quad\leqslant C\frac{(\beta-\alpha+1)m_k}{\rho(y_k,x_k)\rho(y_k,\gamma)} \\
&\quad\leqslant C\frac{(\beta-\alpha+1)m_k}{\rho^2(y_k,\gamma)}.
\end{aligned}$$

The last inequality is implied by the definition of ρ, $y_k\notin 6[\alpha,\beta]$, and $\rho(y_k,x_k)\geqslant\rho(y_k,\gamma)-(\beta-\alpha+1)\geqslant\frac{2}{3}\rho(y_k,\gamma)$. Consequently, we have

$$\begin{aligned}
&\frac{1}{m_k}\sum_{\substack{y_k\in\{0,\dots,m_k-1\}\\ y_k\notin 6[\alpha,\beta]}}\left|\frac{1}{1-r_k(y-x)}-\frac{1}{1-r_k(y-\gamma e_k)}\right| \\
&\quad\leqslant C\sum_{\{y_k:\rho(y_k,\gamma)>3(\beta-\alpha+1)\}}\frac{\beta-\alpha+1}{\rho^2(y_k,\gamma)}\leqslant C.
\end{aligned}$$

In the sequel we consider

$$H_{1,A}f(y)=\left|M_{A-1}\int_{I_{A-1}(y)\setminus I_A(y)}f(x)\frac{1}{1-r_{A-1}(y-x)}\,d\mu(x)\right|,$$

where $y\in G_m\setminus 6I$. This means, that either there exists an $i\leqslant k-1$, such that $y_i\neq z_i$, or $y_0=z_0,\dots,y_{k-1}=z_{k-1}$, and $y_k\notin 6[\alpha,\beta]$.

The case $A>k+1$. In this case

$$I_{A-1}(y)\setminus I_A(y)\subset I_{A-1}(y_0,\dots,y_{A-2})\subset I_{k+1}(y_0,\dots,y_k).$$

If there exists a $i\leqslant k-1$, such that $y_i\neq z_i$, then the sets $I_k(z_0,\dots,z_{k-1})\supset I$, and $I_{k+1}(y_0,\dots,y_k)$ are disjoint. Consequently, $H_{1,A}f(y)=0$.

On the other hand, if $y_0 = z_0, \dots, y_{k-1} = z_{k-1}$, and $y_k \notin 6[\alpha, \beta]$, then the intervals $I_{k+1}(y_0, \dots, y_k) = I_{k+1}(z_0, \dots, z_{k-1}, y_k)$, and $I = \bigcup_{j=\alpha}^{\beta} I_{k+1}(z_0, \dots, z_{k-1}, j)$ are disjoint. Anyway, we have $H_{1,A} f(y) = 0$.

The case $A < k+1$. That is, $A - 1 \leqslant k - 1$. If

$$I_A(y_0, \dots, y_{A-2}, x_{A-1}) \cap I \neq \emptyset,$$

then the condition $y_0 = z_0, \dots, y_{A-2} = z_{A-2}, x_{A-1} = z_{A-1}$ must be fulfilled. It follows that $I \subset I_k(z_0, \dots, z_{k-1}) \subset I_A(y_0, \dots, y_{A-2}, x_{A-1})$. Thus, $I_A(y_0, \dots, y_{A-2}, x_{A-1}) \cap I = I$. Consequently, the function $r_{A-1}(y - x)$ is constant as x ranges over I. This gives

$$\begin{aligned} H_{1,A} f(y) &= \left| M_{A-1} \int_I f(x) \frac{1}{1 - r_{A-1}(y-x)} \, d\mu(x) \right| \\ &= \left| M_{A-1} \frac{1}{1 - r_{A-1}(y - z_{A-1} e_{A-1})} \int_I f(x) \, d\mu(x) \right| = 0. \end{aligned}$$

Consequently, $H_{1,A} f(y)$ may differ from zero only in the case $A = k+1$. It follows

$$H_1 f(y) = \left| M_k \int_{I_{A-1}(y) \setminus I_A(y)} f(x) \frac{1}{1 - r_k(y-x)} \, d\mu(x) \right|.$$

Recall that $y \in G_m \setminus 6I$. Moreover, if $H_{1,A} f(y) \neq 0$, then $y_0 = z_0, \dots, y_{k-1} = z_{k-1}$, and $y_k \notin 6[\alpha, \beta]$. These assumptions give

$$\begin{aligned} &\int_{G_m \setminus 6I} |H_1 f(y)| \, d\mu(y) \\ &\quad = \frac{1}{M_{k+1}} \sum_{\substack{y_k = 0, \dots, m_k - 1 \\ y_k \notin 6[\alpha,\beta]}} \left| M_k \int_I f(x) \frac{1}{1 - r_k(y-x)} \, d\mu(x) \right| \\ &\quad = \frac{1}{M_{k+1}} \sum_{\substack{y_k = 0, \dots, m_k - 1 \\ y_k \notin 6[\alpha,\beta]}} \Bigg| M_k \int_I f(x) \Bigg(\frac{1}{1 - r_k(y-x)} \\ &\qquad - \frac{1}{1 - r_k(y - \gamma e_k)} \Bigg) d\mu(x) \Bigg| \\ &\quad \leqslant \int_I |f(x)| \, d\mu(x) \cdot \frac{1}{m_k} \sum_{\substack{y_k = 0, \dots, m_k - 1 \\ y_k \notin 6[\alpha,\beta]}} \left| \frac{1}{1 - r_k(y-x)} - \frac{1}{1 - r_k(y - \gamma e_k)} \right| \\ &\quad \leqslant C \int_I |f(x)| \, d\mu(x) = C \|f\|_1. \qquad \Box \end{aligned}$$

Since the operator H_1 is of type (L^2, L^2) [13], then by standard argument with the application of Lemma 2.2 it follows that the operator H_1 is of weak type (L^1, L^1). In the book of Bennett and Sharpley [4, pp. 243–249] one can find that an operator of this kind satisfies the following inequality:

$$\|H_1 f\|_1 \leqslant C\|f\|_{L\log^+ L} + C. \tag{1}$$

In other words,

$$\int_{G_m} |H_1 f|\, d\mu \leqslant C \int_{G_m} |f(x)| \log^+(|f(x)|)\, d\mu(x) + C.$$

This inequality plays a prominent role in the proof of the forthcoming lemma, which is the very base of the proof of Theorem 2.1. That is, let function f satisfy (almost) the same properties as in Lemma 2.3, the only difference is that we do not suppose that f has zero meanvalue. That is, f is integrable, and

$$\operatorname{supp} f \subset \bigcup_{j=\alpha}^{\beta} I_k(z,j) =: I.$$

Lemma 2.4.

$$\int_{6I} H_1 f\, d\mu \leqslant C(\|f\|_1 + \|f\|_{L\log^+ L} + \mu(I)).$$

Proof. The shift invariancy of the Haar measure implies that we may assume for the interval I that $z = 0$.

Similarly, we also may suppose that $6[\alpha, \beta] = [0, \delta]$. This means that the union of the intervals $\bigcup_{j \in 6[\alpha,\beta]} I_k(0,j)$ is shifted to the union of intervals $\bigcup_{j \in [0,\delta]} I_k(0,j)$.

Let $y \in 6I = \bigcup_{j \in [0,\delta]} I_k(0,j)$. For $A \leqslant k$ we have $x_{A-1} \neq y_{A-1}$ if $x \in I_{A-1}(y) \backslash I_A(y)$ (see the definition of $H_{1,A} f(y)$). This means that

$$(I_{A-1}(y) \backslash I_A(y)) \cap I = \emptyset,$$

and consequently $H_{1,A} f(y) = 0$.

That is, $A \geqslant k+1$ may be supposed. Consequently,

$$H_1 f = \sup_{A \geqslant k+1} H_{1,A} f \leqslant H_{1,k+1} f + \sup_{A \geqslant k+2} H_{1,A} f =: H_{1,k+1} f + H_1^{\circ} f.$$

Define the Vilenkin group G_m° as follows:

$$G_m^{\circ} := Z_\delta \times \mathop{\times}_{n=k+1}^{\infty} Z_{m_n}.$$

The normalized Haar measure on G_m° is denoted by μ°, $m_0^{\circ} := \delta, m_1^{\circ} := m_{k+1}, \ldots$. For a function $f : G_m \to \mathbb{C}$ we define the function $g : G_m^{\circ} \to \mathbb{C}$ in the following way:

$$g(x) := f(x) \quad (x \in G_m, x_k \in Z_\delta \subset Z_{m_k}, \text{ and } x_0, \ldots x_{k-1} \text{ are fixed}).$$

Then by inequality (1) we have for $\tilde{H}_1$ defined on $L^1(G_m^{\circ})$ in analogous way as H_1 that

$$\int_{G_m^{\circ}} \tilde{H}_1 g\, d\mu^{\circ} \leqslant C \int_{G_m^{\circ}} |g(x)| \log^+(|g(x)|)\, d\mu^{\circ}(x) + C.$$

That is, in other words:

$$\frac{M_{k+1}}{\delta}\int_{\bigcup_{j=0}^{\delta} I_k(0,j)} \sup_{A\geqslant k+2}\left|\delta m_{k+1}\ldots m_{A-2}\frac{M_{k+1}}{\delta}\int_{\bigcup_{j=0}^{\delta} I_k(0,j)\cap(I_{A-1}(y)\setminus I_A(y))} f(x)\right.$$
$$\left.\times\frac{1}{1-r_{A-1}(y-x)}\,d\mu(x)\right|d\mu(y)$$
$$\leqslant C\frac{M_{k+1}}{\delta}\int_{\bigcup_{j=0}^{\delta} I_k(0,j)} |f(x)|\log^+(|f(x)|)\,d\mu(x)+C.$$

This implies (recall that f is zero outside $I=\bigcup_{j\in[\alpha,\beta]} I_k(0,j)$)

$$\int_{\bigcup_{j=0}^{\delta} I_k(0,j)} \sup_{A\geqslant 2}\left|M_{A-1}\int_{I_{A-1}(y)\setminus I_A(y)} f(x)\frac{1}{1-r_{A-1}(y-x)}\,d\mu(x)\right|d\mu(y)$$
$$\leqslant C\int_I |f(x)|\log^+(|f(x)|)\,d\mu(x)+\frac{C\delta}{M_{k+1}}.$$

This follows the inequality

$$\int_{6I} H_1^{\circ} f\leqslant C(\|f\|_{L\log^+ L}+\mu(I)).$$

Finally, it is left to prove the same for $H_{1,k+1}f$. We use a similar method like in the case of H_1°.

By elementary calculus we have for $0\neq|u|\leqslant\pi/2$ that

$$\left|\cot(u)-\frac{1}{u}\right|\leqslant\frac{4|u|}{\pi^2}\leqslant\frac{|u|}{2}.$$

Suppose that $2\delta<m_k$. Since $\delta\geqslant|y_k-x_k|\neq 0$ for $y_k\in[0,\delta]$ and $x_k\in[\alpha,\beta]\subset[0,\delta]$, then we have

$$\left|\cot\left(\pi\frac{y_k-x_k}{2\delta}\right)-\frac{2\delta}{\pi(y_k-x_k)}\right|\leqslant\frac{\pi|y_k-x_k|}{4\delta}$$

and consequently,

$$\left|\cot\left(\pi\frac{y_k-x_k}{m_k}\right)-\frac{m_k}{2\delta}\cot\left(\pi\frac{y_k-x_k}{2\delta}\right)\right|$$
$$\leqslant\left|\cot\left(\pi\frac{y_k-x_k}{m_k}\right)-\frac{m_k}{\pi(y_k-x_k)}\right|+\left|\frac{m_k}{\pi(y_k-x_k)}-\frac{m_k}{2\delta}\cot\left(\pi\frac{y_k-x_k}{2\delta}\right)\right|$$
$$\leqslant\frac{\pi|y_k-x_k|}{2m_k}+\frac{m_k}{2\delta}\frac{\pi|y_k-x_k|}{4\delta}$$
$$\leqslant C\frac{m_k}{\delta}.$$

We give an upper bound for the integral

$$\begin{aligned}
&\int_{6I} H_{1,k+1} f \, d\mu \\
&\leqslant \frac{1}{2} \int_{\bigcup_{j=0}^{\delta} I_k(0,j)} \left| M_k \int_{\bigcup_{x_k \in [\alpha,\beta] \setminus \{y_k\}} I_{k+1}(0,\dots,0,x_k)} f(x) \, d\mu(x) \right| d\mu(y) \\
&\quad + \frac{1}{2} \int_{\bigcup_{j=0}^{\delta} I_k(0,j)} \left| M_k \frac{m_k}{2\delta} \int_{\bigcup_{x_k \in [\alpha,\beta] \setminus \{y_k\}} I_{k+1}(0,\dots,0,x_k)} f(x) \right. \\
&\quad \left. \times \cot\left(\pi \frac{y_k - x_k}{2\delta}\right) d\mu(x) \right| d\mu(y) \\
&\quad + C \frac{1}{2} \int_{\bigcup_{j=0}^{\delta} I_k(0,j)} M_k \frac{m_k}{2\delta} \int_{\bigcup_{x_k \in [\alpha,\beta] \setminus \{y_k\}} I_{k+1}(0,\dots,0,x_k)} |f(x)| \, d\mu(x) \, d\mu(y) \\
&=: H_{1,k+1,1} + H_{1,k+1,2} + H_{1,k+1,3}.
\end{aligned}$$

It is easy to check the first and the third addend:

$$H_{1,k+1,1} + H_{1,k+1,3} \leqslant C\mu\left(\bigcup_{j=0}^{\delta} I_k(0,j) \right) \frac{M_{k+1}}{\delta} ||f||_1 = C||f||_1.$$

Since $\frac{1}{1-\exp(2\pi\imath(y_k - x_k)/(2\delta))} = 1/2 + \imath/2 \cot(\pi(y_k - x_k)/(2\delta))$, then we have to investigate

$$\int_{\bigcup_{j=0}^{\delta} I_k(0,j)} \left| \frac{M_{k+1}}{2\delta} \int_{\bigcup_{x_k \in [\alpha,\beta] \setminus \{y_k\}} I_{k+1}(0,\dots,0,x_k)} f(x) \frac{1}{1 - \exp\left(2\pi\imath \frac{y_k - x_k}{2\delta}\right)} d\mu(x) \right| d\mu(y).$$

The distance of this integral and $H_{1,k+1,2}$ is bounded by $C||f||_1$.

By (1) for the Vilenkin group $G_m^\dagger$

$$G_m^\dagger := Z_{2\delta} \times \mathop{\times}_{n=k+1}^{\infty} Z_{m_n},$$

we have

$$\begin{aligned}
&\frac{M_{k+1}}{2\delta} \int_{\bigcup_{j=0}^{\delta} I_k(0,j)} \left| \frac{M_{k+1}}{2\delta} \int_{\bigcup_{l=0, l \neq j}^{\delta} I_{k+1}(0,\dots,0,l)} f(x) \frac{1}{1 - \exp\left(2\pi\imath \frac{j-l}{2\delta}\right)} d\mu(x) \right| d\mu(y) \\
&\leqslant C \frac{M_{k+1}}{2\delta} \int_{\bigcup_{j=0}^{\delta} I_k(0,j)} |f(x)| \log^+(|f(x)|) \, d\mu(x) + C.
\end{aligned}$$

This gives

$$\int_{\bigcup_{j=0}^{\delta} I_k(0,j)} \left| \frac{M_{k+1}}{2\delta} \int_{\bigcup_{x_k \in [\alpha,\beta]\setminus\{y_k\}} I_{k+1}(0,\dots,0,x_k)} f(x) \frac{1}{1-\exp\left(2\pi\imath \frac{y_k - x_k}{2\delta}\right)} \, d\mu(x) \right| d\mu(y)$$
$$\leqslant C(\|f\|_{L\log^+ L} + \mu(I)).$$

Consequently, we have $\int_{6I} H_{1,k+1} f \, d\mu \leqslant C(\|f\|_{L\log^+ L} + \|f\|_1 + \mu(I))$.

What can be said for $H_{1,k+1}f$ if $m_k \leqslant 2\delta$? Recall the definition of $6[\alpha,\beta]$ before Lemma 2.3, and the definition of δ at the beginning of the proof of this lemma. That is, if $2\delta < m_k$ is not fulfilled, then $6[\alpha,\beta] = [0,\delta] = [0, m_k - 1]$. The method is the same, except the point that we do not interchange $\exp(2\pi\imath(y_k - x_k)/m_k)$ by $\exp(2\pi\imath(y_k - x_k)/(2\delta))$, that is we take $G_m^\dagger := Z_{m_k} \times \times_{n=k+1}^{\infty} Z_{m_n}$ not $Z_{2\delta} \times \times_{n=k+1}^{\infty} Z_{m_n}$. □

Remark 2.5. If the integral of function f equals with zero, then by Lemmas 2.3 and 2.4 we have $\|H_1 f\|_1 \leqslant C(\|f\|_1 + \|f\|_{L\log^+ L} + \mu(I))$.

In the sequel we turn our attention to the two-dimensional case. Define the two-dimensional operators $\mathbf{L}_{1,1}, \mathbf{H}_{1,1}$ for integrable functions in the following way.

$$\mathbf{L}_{1,1,A,B} f(y^1, y^2) := (L_{1,A} \otimes L_{1,B}) f(y^1, y^2),$$

$$\mathbf{H}_{1,1,A,B} f := |\mathbf{L}_{1,1,A,B} f|,$$

$$\mathbf{H}_{1,1} f := \sup_{A,B,|A-B|<\alpha} \mathbf{H}_{1,1,A,B} f$$

for some fixed $\alpha > 0$. We discuss the two-dimensional operator $\mathbf{H}_{1,1}$ with respect to functions supported by squares. Let f be an integrable two-parameter function on the group $G_m \times G_{\tilde{m}}$, and

$$\int_{G_m} f(x^1, x^2) \, d\mu(x^1) = \int_{G_{\tilde{m}}} f(x^1, x^2) \mu(x^2) = 0$$

for all $x^2 \in G_{\tilde{m}}$ and $x^1 \in G_m$. Let the support of f be included in the square below:

$$\operatorname{supp} f \subset \bigcup_{j^1=\alpha^1}^{\beta^1} I_k(z^1, j^1) \times \bigcup_{j^2=\alpha^2}^{\beta^2} I_k(z^2, j^2) \times =: I^1 \times I^2 =: I.$$

Use the notation $6I := 6I^1 \times 6I^2$. We prove the two-dimensional version of Lemma 2.3.

Lemma 2.6.

$$\int_{G_m \times G_{\tilde{m}} \setminus 6I} \mathbf{H}_{1,1} f \leqslant C\alpha(\|f\|_1 + \|f\|_{L\log^+ L} + \mu(I)).$$

Proof. The set $G_m \times G_{\tilde{m}} \backslash 6I$ is the disjoint union of the sets $(G_m \backslash 6I^1) \times (G_{\tilde{m}} \backslash 6I^2) =: J_1$, $6I^1 \times (G_{\tilde{m}} \backslash 6I^2) =: J_2$, and $(G_m \backslash 6I^1) \times 6I^2 =: J_3$. We discuss the integrals of the function $\mathbf{H}_{1,1} f$ on these sets one by one. First on J_1, then on J_2. Since the case J_3 is similar like case J_2 (interchanging the variables), then it is left to the reader. In order to have the necessary bound for the integral on J_1 we apply the method of Lemma 2.3.

That is, let $(y^1, y^2) \in J_1$. Consequently, $y^1 \notin 6I^1$, and $y^2 \notin 6I^2$. This implies that $(L_{1,A} \otimes L_{1,B}) f(y^1, y^2)$ may be different from zero only in the case, when $A = k+1$, and $B = k+1$. Besides, $y^1 \in I_k(z^1), y^2 \in I_k(z^2)$, and $y^1 \notin 6[\alpha^1, \beta^1]$, $y^2 \notin 6[\alpha^2, \beta^2]$. These assumptions give

$$
\begin{aligned}
\mathbf{H}_{1,1} f(y^1, y^2) &= |\mathbf{L}_{1,1,k+1,k+1} f(y^1, y^2)| \\
&= \left| M_k \tilde{M}_k \int_{I^1 \times I^2} f(x^1, x^2) \frac{1}{1 - r_k(y^1 - x^1)} \right. \\
&\quad \left. \times \frac{1}{1 - r_k(y^2 - x^2)} \, d\mu(x^1, x^2) \right| \\
&= \left| M_k \tilde{M}_k \int_{I^1 \times I^2} f(x^1, x^2) \left(\frac{1}{1 - r_k(y^1 - x^1)} - \frac{1}{1 - r_k(y^1 - \gamma^1 e_k)} \right) \right. \\
&\quad \left. \times \left(\frac{1}{1 - r_k(y^2 - x^2)} - \frac{1}{1 - r_k(y^2 - \gamma^2 e_k)} \right) d\mu(x^1, x^2) \right| \\
&\leqslant C \frac{\beta^1 - \alpha^1 + 1}{\rho^2(y^1_k, \gamma^1)} \frac{\beta^2 - \alpha^2 + 1}{\rho^2(y^2_k, \gamma^2)} M_{k+1} \tilde{M}_{k+1} \int_{I^1 \times I^2} |f(x^1, x^2)| \, d\mu(x^1, x^2).
\end{aligned}
$$

So,

$$
\mathbf{H}_{1,1} f(y^1, y^2) \leqslant C \frac{\beta^1 - \alpha^1 + 1}{\rho^2(y^1_k, \gamma^1)} \frac{\beta^2 - \alpha^2 + 1}{\rho^2(y^2_k, \gamma^2)} M_{k+1} \tilde{M}_{k+1} ||f||_1.
$$

At this situation we apply Lemma 2.3, or more exactly, the method of the proof, that is, that fact that $\sum_{y^j_k \notin 6[\alpha^j, \beta^j]} \frac{1}{\rho^2(y^j_k, \gamma^j)} \leqslant C \frac{1}{\beta^j - \alpha^j + 1}$ $(j = 1, 2)$. This gives the inequality:

$$
\int_{J_1} \mathbf{H}_{1,1} f(y^1, y^2) \, d\mu(y^1, y^2) \leqslant C ||f||_1.
$$

Let now, $y \in J_2$, that is, $y^1 \in 6I^1, y^2 \notin 6I^2$. This implies that $(L_{1,A} \otimes L_{1,B}) f(y^1, y^2)$ may be different from zero only in the case, when $B = k+1$ and so $k + 1 - \alpha \leqslant A \leqslant k + 1 + \alpha$. Besides, $y^2 \in I_k(z^2)$, and $y^2_k \notin 6[\alpha^2, \beta^2]$. This gives

$$
\begin{aligned}
&|\mathbf{L}_{1,1,A,k+1} f(y^1, y^2)| \\
&\quad = \left| \tilde{M}_k \int_{I^2} L_{1,A} f(y^1, x^2) \left(\frac{1}{1 - r_k(y^2 - x^2)} - \frac{1}{1 - r_k(y^2 - \gamma^2 e_k)} \right) d\mu(x^2) \right| \\
&\quad \leqslant C \frac{\tilde{M}_{k+1}(\beta^2 - \alpha^2 + 1)}{\rho^2(y^2, \gamma^2)} \int_{I^2} |L_{1,A} f(y^1, x^2)| \, d\mu(x^2).
\end{aligned}
$$

This gives

$$\begin{aligned}
&\int_{6I^1\times(G_{\tilde m}\setminus 6I^2)} |\mathbf{L}_{1,1,A,k+1} f(y^1,y^2)|\,d\mu(y^1,y^2)\\
&\leqslant \int_{6I^1} \frac{1}{\tilde M_{k+1}} \sum_{y_k^2\notin 6[\alpha^2,\beta^2]} \frac{\tilde M_{k+1}(\beta^2-\alpha^2+1)}{\rho^2(y_k^2,\gamma^2)} \int_{I^2} |L_{1,A} f(y^1,x^2)|\,d\mu(x^2)\,d\mu(y^1)\\
&\leqslant C\int_{6I^1}\int_{I^2} |L_{1,A} f(y^1,x^2)|\,d\mu(x^2)d\mu(y^1)\\
&= C\int_{I^2}\int_{6I^1} |L_{1,A} f(y^1,x^2)|\,d\mu(y^1)d\mu(x^2).
\end{aligned}$$

Apply Lemma 2.4 for each fixed x^2. This gives

$$\begin{aligned}
&\int_{I^2}\int_{6I^1} |L_{1,A} f(y^1,x^2)|\,d\mu(y^1)\,d\mu(x^2)\\
&\leqslant C\int_{I^2}\Bigg(\int_{I^1} |f(x^1,x^2)|\,d\mu(x^1)\\
&\quad + \int_{I^1} |f(x^1,x^2)|\log^+(|f(x^1,x^2)|)\,d\mu(x^1) + \mu(I^1)\Bigg) d\mu(x^2)\\
&\leqslant C(\|f\|_1 + \|f\|_{L\log^+ L} + \mu(I)).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_{J_2} |\mathbf{H}_{1,1} f(y^1,y^2)|\,d\mu(y^1,y^2) &\leqslant \sum_{A=k+1-\alpha}^{A=k+1+\alpha} \int_{J_2} |\mathbf{L}_{1,1,A,k+1} f(y^1,y^2)|\,d\mu(y^1,y^2)\\
&\leqslant C\alpha(\|f\|_1 + \|f\|_{L\log^+ L} + \mu(I)).
\end{aligned}$$

The proof of Lemma 2.6 is complete. □

Recall the notation of Lemma 2.6, and for $j\in\mathbb{N}$ let $g_j : G_m\times G_{\tilde m}\to\mathbb{C}$ be integrable with the property

$$g_j(x^1,x^2) = \mu^{-1}(I_j^1) 1_{I_j^1}(x^1) \int_{I_j^1} f_j(t^1,x^2)\,d\mu(t^1),$$

where $f, f_j : G_m\times G_{\tilde m}\to\mathbb{C}$ are also integrable, and

$$\operatorname{supp} f_j \subset I_j = I_j^1\times I_j^2,\quad \int_{G_m\times G_{\tilde m}} f_j\,d\mu = 0.$$

Moreover, the two-dimensional rectangles I_j are disjoint, and $f=\sum_{j\in\mathbb{N}} f_j,\ g=\sum_{j\in\mathbb{N}} g_j,\ I=\bigcup_{j\in\mathbb{N}} I_j$.

Lemma 2.7.

$$\mu(\mathbf{H}_{1,1,A,B}g>\lambda)\leqslant\frac{C}{\lambda}(\|f\|_1+\|f\|_{L\log^+L}+\mu(I))$$

for any $\lambda>0$ *and* $A,B\in\mathbb{N}$.

Proof. The proof is quite simple and based on the fact that the one-dimensional operator H_1 (and so $H_{1,A}$) is of weak type $(1,1)$, and on the inequality of Lemma 2.3. Fix an $y^2\in G_{\tilde{m}}$. Then the one-dimensional Haar measure of the set

$$\{y^1\in G_m:\ \mathbf{H}_{1,1,A,B}g(y^1,y^2)>\lambda\}$$

is bounded by

$$\frac{C}{\lambda}\int_{G_m}|L_{1,B}g(y^1,y^2)|\,d\mu(y^1).$$

This follows from the fact that $H_{1,A}$ is of weak type $(1,1)$. Consequently, the two-dimensional measure

$$\begin{aligned}\mu(\mathbf{H}_{1,1,A,B}g>\lambda)&\leqslant\frac{C}{\lambda}\int_{G_{\tilde{m}}}\int_{G_m}|L_{1,B}g(y^1,y^2)|\,d\mu(y^1)\,d\mu(y^2)\\&=\sum_{j\in\mathbb{N}}\frac{C}{\lambda}\int_{G_{\tilde{m}}}\int_{G_m}\mu^{-1}(I_j^1)1_{I_j^1}(y^1)|L_{1,B}\tilde{g}_j(y^2)|\,d\mu(y^1)\,d\mu(y^2),\end{aligned}$$

where $\tilde{g}_j(x^2)=\int_{I_j^1}f_j(t^1,x^2)\,d\mu(t^1)$. On the other hand,

$$L_{1,B}\tilde{g}_j(.)=\int_{I_j^1}L_{1,B}f_j(t^1,.)\,d\mu(t^1),$$

that is, the operator $L_{1,B}$ with respect to the second variable of f_j, and the integral on the set I_j^1 with respect to the first variable are interchangeable. Thus, by Lemmas 2.3 and 2.4 we have

$$\begin{aligned}&\mu(\mathbf{H}_{1,1,A,B}g>\lambda)\\&\quad\leqslant\frac{C}{\lambda}\sum_{j\in\mathbb{N}}\int_{G_{\tilde{m}}}|L_{1,B}\tilde{g}_j(y^2)|\,d\mu(y^2)\\&\quad\leqslant\frac{C}{\lambda}\sum_{j\in\mathbb{N}}\left(\int_{G_{\tilde{m}}\setminus 6I_j^2}|L_{1,B}\tilde{g}_j(y^2)|\,d\mu(y^2)+\frac{C}{\lambda}\int_{6I_j^2}|L_{1,B}\tilde{g}_j(y^2)|\,d\mu(y^2)\right)\\&\quad\leqslant\frac{C}{\lambda}\sum_{j\in\mathbb{N}}\left(\|\tilde{g}_j\|_1+\int_{6I_j^2}\int_{I_j^1}|L_{1,B}f_j(t^1,y^2)|\,d\mu(t^1)\,d\mu(y^2)\right)\end{aligned}$$

$$= \frac{C}{\lambda}\sum_{j\in\mathbb{N}}\left(\|f_j\|_1 + \int_{I_j^1}\int_{6I_j^2} |L_{1,B} f_j(t^1,y^2)| d\mu(y^2)\, d\mu(t^1)\right)$$
$$\leqslant \frac{C}{\lambda}\sum_{j\in\mathbb{N}}\left(\|f_j\|_1 + \int_{I_j^1}\left(\int_{G_{\tilde m}} |f_j(t^1,x^2)|\, d\mu(x^2)\right.\right.$$
$$\left.\left. + \int_{G_{\tilde m}} |f_j(t^1,x^2)|\log^+(|f_j(t^1,x^2)|)\, d\mu(x^2) + \mu(I_j^2)\right) d\mu(t^1)\right)$$
$$\leqslant \frac{C}{\lambda}\sum_{j\in\mathbb{N}} (\|f_j\|_1 + \|f_j\|_{L\log^+ L} + \mu(I_j))$$
$$\leqslant \frac{C}{\lambda}(\|f\|_1 + \|f\|_{L\log^+ L} + \mu(I)).$$

This completes the proof of this lemma. □

Next, we need the two-dimensional version of Lemma 2.2.

Lemma 2.8. *Let $f\in L^1(G_m\times G_{\tilde m})$, and $\lambda > \|f\|_1 > 0$ arbitrary. Then the function f can be decomposed in the following form*:

$$f = f_0 + \sum_{j=1}^{\infty} f_j, \quad \|f_0\|_\infty \leqslant C\lambda, \quad \|f_0\|_1 \leqslant C\|f\|_1,$$

$$\operatorname{supp} f_j \subset \bigcup_{l=\alpha_j^1}^{\beta_j^1} I_{k_j}(z^{1,j},l) \times \bigcup_{l=\alpha_j^2}^{\beta_j^2} I_{k_j}(z^{2,j},l) = I_j^1\times I_j^2 = J_j, \quad \int_{G_m\times G_{\tilde m}} f_j\, d\mu = 0$$
$$(j\in\mathbb{P})$$

and for

$$F = \bigcup_{j\in\mathbb{P}} J_j, \quad \mu(F) \leqslant C\frac{\|f\|_1}{\lambda}.$$

Moreover, *the sets J_j* (*we can call them squares*) *are disjoint* $(j\in\mathbb{P})$.

We prove that the maximal operator $\mathbf{H}_{1,1}$ is of weak type $(L\log^+ L, L^1)$ "almost", or more exactly we prove:

Lemma 2.9. *For all integrable function $f\in L\log^+ L(G_m\times G_{\tilde m})$ we have*

$$\mu(\mathbf{H}_{1,1} f > \lambda) \leqslant \frac{C}{\lambda}\alpha^2(\|f\|_1 + \|f\|_{L\log^+ L}) + \frac{C}{\lambda^2}\alpha^2\|f\|_1$$

for any $\lambda > 0$.

Proof. Apply the two-dimensional Calderon–Zygmund decomposition lemma, that is, Lemma 2.8.

$$\begin{aligned}
f &= f_0 + \sum_{j=1}^{\infty}\left(f_j - \mu^{-1}(I_j^1)1_{I_j^1}\int_{I_j^1} f_j - \mu^{-1}(I_j^2)1_{I_j^2}\int_{I_j^2} f_j\right)\\
&\quad + \sum_{j=1}^{\infty}\mu^{-1}(I_j^1)1_{I_j^1}\int_{I_j^1} f_j + \sum_{j=1}^{\infty}\mu^{-1}(I_j^2)1_{I_j^2}\int_{I_j^2} f_j\\
&=: f_0 + \sum_{j=1}^{\infty} f_j' + \sum_{j=1}^{\infty} g_j^1 + \sum_{j=1}^{\infty} g_j^2.
\end{aligned}$$

By the sublinearity of the operator $\mathbf{H}_{1,1}$ we have

$$\begin{aligned}
&\mu(\mathbf{H}_{1,1} f > \lambda)\\
&\quad \leqslant \mu(\mathbf{H}_{1,1} f_0 > \lambda/2) + \mu(6F) + \mu\left(\overline{6F}\cap\left(\mathbf{H}_{1,1}\left(\sum_{j=1}^{\infty} f_j'\right) > \lambda/6\right)\right)\\
&\qquad + \mu\left(\overline{6F}\cap\left(\mathbf{H}_{1,1}\left(\sum_{j=1}^{\infty} g_j^1\right) > \lambda/6\right)\right) + \mu\left(\overline{6F}\cap\left(\mathbf{H}_{1,1}\left(\sum_{j=1}^{\infty} g_j^2\right) > \lambda/6\right)\right)\\
&\quad =: v_1 + v_2 + v_3 + v_4 + v_5.
\end{aligned}$$

Since the one-dimensional operator H_1 is of type (L^2, L^2), then so does the two-dimensional version $\mathbf{H}_{1,1}$. That is,

$$v_1 \leqslant \frac{C}{\lambda^2}\|\mathbf{H}_{1,1} f_0\|_2^2 \leqslant \frac{C}{\lambda^2}\|f_0\|_2^2 \leqslant \frac{C}{\lambda}\|f_0\|_1 \leqslant \frac{C}{\lambda}\|f\|_1$$

as it follows from Lemma 2.8. This lemma also implies $v_2 \leqslant \frac{C}{\lambda}\|f\|_1$.

Next, we discuss v_4. Its investigation based on Lemma 2.7.

$$\begin{aligned}
v_4 &\leqslant \mu\left(\overline{6F}\cap\bigcup_{A=1}^{\infty}\bigcup_{B=A-\alpha}^{B=A+\alpha}\left(\mathbf{H}_{1,1,A,B}\left(\sum_{j=1}^{\infty} g_j^1\right) > \lambda/6\right)\right)\\
&\leqslant \sum_{A=1}^{\infty}\sum_{B=A-\alpha}^{B=A+\alpha}\mu\left(\overline{6F}\cap\left(\mathbf{H}_{1,1,A,B}\left(\sum_{j=1}^{\infty} g_j^1\right) > \lambda/6\right)\right).
\end{aligned}$$

In order to get a bound for v_4 we prove that there are only at most α number of k_j's such that $\mathbf{H}_{1,1,A,B} g_j^1$ differs from zero on the complementer set of $6F$. In the proof of Lemma 2.3 one can find that for any $B \leqslant k_j$ the function

$$\begin{aligned}
&L_{1,B}\left(\mu^{-1}(I_j^1)1_{I_j^1}(y^1)\int_{I_j^1} f_j(t^1,.)\,d\mu(t^1)\right)(y^2)\\
&\quad = \mu^{-1}(I_j^1)1_{I_j^1}(y^1)L_{1,B}\left(\int_{\bigcup_{l=\alpha_j^1}^{\beta_j^1} I_{k_j}(z^{1,j,l})} f_j(t^1,.)\,d\mu(t^1)\right)(y^2) = 0
\end{aligned}$$

on every $y = (y^1, y^2) \in G_m \times G_{\tilde{m}}$ because the integral of the function $\int_{I_j^1} f_j(t^1, .)\, d\mu(t^1)$ is zero. That is, $B > k_j$ can be supposed. Since for $y \notin 6F$ we have $y \notin 6J_j = 6I_j^1 \times 6I_j^2$, then either $y^2 \notin 6I_j^2$, or $y^1 \notin 6I_j^1$.

In the first case, also in the proof of Lemma 2.3 one can find that

$$L_{1,B}\left(\int_{\bigcup_{l=\alpha_j^1}^{\beta_j^1} I_{k_j}(z^{1,j},l)} f_j(t^1, .)\, d\mu(t^1)\right)(y^2)$$

may be different from zero only in the case $B - 1 = k_j$. That is, we have only one k_j such that $\mathbf{H}_{1,1,A,B} g_j^1$ differs from zero.

See the second case, that is, $y^1 \notin 6I_j^1$. This follows $y^1 \notin I_j^1$, that is, $y^1 \notin \bigcup_{l=\alpha_j^1}^{\beta_j^1} I_{k_j}(z^{1,j}, l)$, and consequently, $D_{M_A}(y^1 - x^1) = 0$ for $x^1 \in I_j^1$, and $A \geqslant k_j + 1$. The definition of $L_{1,A}$ gives that $L_{1,A} g_j^1(y) = 0$. This means, that $A \leqslant k_j < B$ is fulfilled. Since the distance between A and B is less than α, then we proved that there are only at most α number of k_j's such that $\mathbf{H}_{1,1,A,B} g_j^1$ differs from zero on the complementer set of $6F$.

So, by the help of Lemma 2.7 we may continue the investigation of v_4.

$$\begin{aligned}
v_4 &\leqslant \sum_{A=1}^{\infty} \sum_{B=A-\alpha}^{B=A+\alpha} \mu\left(\overline{6F} \cap \left(\mathbf{H}_{1,1,A,B}\left(\sum_{j=1}^{\infty} g_j^1\right) > \lambda/6\right)\right) \\
&= \sum_{A=1}^{\infty} \sum_{B=A-\alpha}^{B=A+\alpha} \mu\left(\overline{6F} \cap \left(\mathbf{H}_{1,1,A,B}\left(\sum_{\{j\,:\,A \leqslant k_j < A+\alpha\}} g_j^1\right) > \lambda/6\right)\right) \\
&\leqslant \frac{C}{\lambda} \sum_{A=1}^{\infty} \sum_{B=A-\alpha}^{B=A+\alpha} \sum_{\{j\,:\,A \leqslant k_j < A+\alpha\}} (\|f_j\|_1 + \|f_j\|_{L \log^+ L} + \mu(J_j)) \\
&\leqslant \frac{C}{\lambda} \alpha \sum_{A=1}^{\infty} \sum_{\{j\,:\,A \leqslant k_j < A+\alpha\}} (\|f_j\|_1 + \|f_j\|_{L \log^+ L} + \mu(J_j)) \\
&\leqslant \frac{C}{\lambda} \alpha^2 (\|f\|_1 + \|f\|_{L \log^+ L} + \mu(F)) \\
&\leqslant \frac{C}{\lambda} \alpha^2 (\|f\|_1 + \|f\|_{L \log^+ L}) + \frac{C}{\lambda^2} \alpha^2 \|f\|_1.
\end{aligned}$$

The investigate v_5 is the same procedure as v_4, that is why, it is left to the reader.

The rest is to find an appropriate bound for v_3. Basically, the proof of nothing else but the application of Lemma 2.6. Therefore,

$$v_3 = \mu\left(\overline{6F} \cap \left(\mathbf{H}_{1,1}\left(\sum_{j=1}^{\infty} f_j'\right) > \lambda/6\right)\right) \leqslant \frac{C}{\lambda} \int_{\overline{6F}} \mathbf{H}_{1,1}\left(\sum_{j=1}^{\infty} f_j.\right) d\mu$$

$$\leqslant \frac{C}{\lambda} \sum_{j=1}^{\infty} \int_{\overline{6J_j}} \mathbf{H}_{1,1} f_j' \, d\mu$$

$$\frac{C}{\lambda} \sum_{j=1}^{\infty} (||f_j'||_1 + ||f_j'||_{L \log^+ L} + \mu(J_j)).$$

By elementary calculus we can prove the inequality

$$|u+v|\log^+(|u+v|) \leqslant |u|\log^+(|u|) + |v|\log^+(|v|) + |u| + |v|$$

for any $u, v \in \mathbb{C}$ complex numbers. This implies $||g+h||_{L \log^+ L} \leqslant ||g||_{L \log^+ L} + ||h||_{L \log^+ L} + ||g||_1 + ||h||_1$ for every integrable function g, h. Consequently,

$$||f_j'||_{L \log^+ L} \leqslant ||f_j||_{L \log^+ L} + ||g_j^1||_{L \log^+ L} + ||g_j^2||_{L \log^+ L} + 2(||f_j||_1 + ||g_j^1||_1 + ||g_j^2||_1).$$

On the other hand,

$$||g_j^1||_1 \leqslant \int_{G_{\tilde{m}}} \int_{G_m} \mu^{-1}(I_j^1) 1_{I_j^1}(x^1) \int_{I_j^1} |f(t^1, x^2)| \, d\mu(t^1) \, d\mu(x^1) \, d\mu(x^2) = ||f_j||_1.$$

The function $|u|\log^+(|u|)$ is convex on $[0, +\infty)$. This implies

$$\begin{aligned} ||g_j^1||_{L \log^+ L} &= \int_{G_{\tilde{m}}} \int_{I_j^1} \mu^{-1}(I_j^1) \left| \int_{I_j^1} f_j(t^1, x^2) \, d\mu(t^1) \right| \\ &\quad \times \log^+ \left(\mu^{-1}(I_j^1) \left| \int_{I_j^1} f_j(t^1, x^2) \, d\mu(t^1) \right| \right) d\mu(x^1) \, d\mu(x^2) \\ &\leqslant \int_{G_{\tilde{m}}} \int_{I_j^1} |f_j(x^1, x^2)|\log^+(|f_j(x^1, x^2)|) \, d\mu(x^1) \, d\mu(x^2) \\ &= ||f_j||_{L \log^+ L}. \end{aligned}$$

That is, since the support of the functions f_j are disjoint, then

$$\begin{aligned} v_3 &\leqslant \frac{C}{\lambda} \sum_{j=1}^{\infty} (||f_j||_1 + ||f_j||_{L \log^+ L} + \mu(J_j)) \\ &\leqslant \frac{C}{\lambda}(||f||_1 + ||f||_{L \log^+ L}) + \frac{C}{\lambda} \mu(F) \\ &\leqslant \frac{C}{\lambda}(||f||_1 + ||f||_{L \log^+ L}) + \frac{C}{\lambda^2} ||f||_1. \end{aligned}$$

This completes the proof of Lemma 2.9. □

Define the two-dimensional operators $\mathbf{F}_1^i$ $(i = 1, 2)$ for integrable functions in the following way:

$$\mathbf{F}_{1,A,B}^1 f(y^1, y^2) := (E_A^1 \otimes L_{1,B}) f(y^1, y^2), \quad \mathbf{F}_{1,A,B}^2 f(y^1, y^2) := (L_{1,A} \otimes E_B^2) f(y^1, y^2),$$

$$\mathbf{F}_1^i f := \sup_{A,B,|A-B|<\alpha} |\mathbf{F}_{1,A,B}^i f|,$$

where $i = 1, 2$. We prove some kind of weak boundedness of the operators $\mathbf{F}_1^i$.

Lemma 2.10. $\mu(\mathbf{F}_1^i f > \lambda) \leqslant \frac{C}{\lambda}(\alpha^2 ||f||_1 + ||f||_{L\log^+ L}) + \frac{C}{\lambda^2}||f||_1$.

Proof. Since the cases $i = 1$ and 2 are quite similar, we discuss the case $i = 1$, only. Apply the notation of the proof of Lemma 2.9. That is,

$$f = f_0 + \sum_{j=1}^{\infty} f_j' + \sum_{j=1}^{\infty} g_j^1 + \sum_{j=1}^{\infty} g_j^2.$$

It is well known that the one-dimensional maximal operator $h^* := \sup_A |E_A h|$ is of weak type $(1,1)$ also on unbounded Vilenkin groups. Apply this fact, and Remark 2.5. We recall that for all fixed $x^1 \in I_j^1$ the integral $\int_{G_{\tilde{m}}} f_j'(x^1, x^2)\, d\mu(x^2)$ is zero.

$$\begin{aligned}
&\mu\left(\mathbf{F}_1^1\left(\sum_{j=1}^{\infty} f_j'\right) > \lambda\right) \\
&\leqslant \mu\left(\left(\sup_B \left|L_{1,B}\left(\sum_{j=1}^{\infty} f_j'\right)\right|\right)^* > \lambda\right) \\
&\leqslant \frac{C}{\lambda}\int_{G_m}\int_{G_{\tilde{m}}} \sup_B \left|L_{1,B}\left(\sum_{j=1}^{\infty} f_j'\right)(x^1, y^2)\right| d\mu(y^2)\, d\mu(x^1) \\
&\leqslant \frac{C}{\lambda}\sum_{j=1}^{\infty}\int_{G_m}\int_{G_{\tilde{m}}} \sup_B |L_{1,B} f_j'(x^1, y^2)|\, d\mu(y^2)\, d\mu(x^1) \\
&\leqslant \frac{C}{\lambda}\sum_{j=1}^{\infty}\int_{I_j^1}\int_{G_{\tilde{m}}} \sup_B |L_{1,B} f_j'(x^1, y^2)|\, d\mu(y^2)\, d\mu(x^1) \\
&\leqslant \frac{C}{\lambda}\sum_{j=1}^{\infty}\left(\int_{I_j^1}\left(\int_{G_{\tilde{m}}} |f_j'(x^1, x^2)|\, d\mu(x^2)\right.\right. \\
&\quad \left.\left. + \int_{G_{\tilde{m}}} |f_j'(x^1, x^2)| \log(|f_j'(x^1, x^2)|)\, d\mu(x^2) + \mu(I_j^2)\right) d\mu(x^1)\right) \\
&\leqslant \frac{C}{\lambda}\sum_{j=1}^{\infty} (||f_j'||_1 + ||f_j'||_{L\log^+ L} + \mu(J_j)).
\end{aligned}$$

At the end of the proof of Lemma 2.9 one can find the inequalities

$$\begin{aligned}
||f_j'||_{L\log^+ L} &\leqslant ||f_j||_{L\log^+ L} + ||g_j^1||_{L\log^+ L} + ||g_j^2||_{L\log^+ L} \\
&\quad + 2(||f_j||_1 + ||g_j^1||_1 + ||g_j^2||_1) \\
&\leqslant C(||f_j||_1 + ||f_j||_{L\log^+ L}).
\end{aligned}$$

Consequently, the two-dimensional Calderon–Zygmund lemma, that is, Lemma 2.8 gives

$$\mu\left(\mathbf{F}_1^1\left(\sum_{j=1}^{\infty} f_j'\right) > \lambda\right) \leqslant \frac{C}{\lambda}(\|f\|_1 + \|f\|_{L\log^+ L}) + \frac{C}{\lambda^2}\|f\|_1. \tag{2}$$

Fix an $A, B \in \mathbb{N}$. The one-dimensional operator H_1 is of weak type $(1,1)$, and the one-dimensional operator E_A is of type (L^1, L^1). This gives

$$\begin{aligned}\mu(|(E_A^1 \otimes L_{1,B})f| > \lambda) &= \mu(|L_{1,B}(E_A^1 f)| > \lambda)\\ &\leqslant \frac{C}{\lambda}\int_{G_m}\int_{G_{\tilde{m}}} E_A^1 f(y^1, x^2)\, d\mu(x^2)\, d\mu(y^1) \leqslant \frac{C}{\lambda}\|f\|_1. \end{aligned}\tag{3}$$

In the sequel we apply the method used in the proof of Lemma 2.9 at the point we got bound for v_4. That is, we prove that there are only at most α number of k_j's such that $\mathbf{F}_{1,A,B}^1 g_j^1$ differs from zero on the complementer set of $6F$.

In the proof of Lemma 2.3 one can find that for any $B \leqslant k_j$ the function

$$\begin{aligned} L_{1,B} g_j^1(x^1, y^2) &= L_{1,B}\left(\mu^{-1}(I_j^1) 1_{I_j^1}(x^1) \int_{I_j^1} f(t^1, .)\, d\mu(t^1)\right)(y^2)\\ &= \mu^{-1}(I_j^1) 1_{I_j^1}(x^1) L_{1,B}\left(\int_{\bigcup_{l=\alpha_j^1}^{\beta_j^1} I_{k_j}(z^{1,j}, l)} f(t^1, .)\, d\mu(t^1)\right)(y^2) = 0 \end{aligned}$$

on every $y = (y^1, y^2) \in G_m \times G_{\tilde{m}}$ because the integral of the function $\int_{I_j^1} f(t^1, .)\, d\mu(t^1)$ (the support of which is a subset of I_j^2) is zero. That is, $B > k_j$ can be supposed. Since for $y \notin 6F$ we have $y \notin 6J_j = 6I_j^1 \times 6I_j^2$, then either $y^2 \notin 6I_j^2$, or $y^1 \notin 6I_j^1$.

In the first case, also in the proof of Lemma 2.3 one can find that

$$L_{1,B}\left(\int_{\bigcup_{l=\alpha_j^1}^{\beta_j^1} I_{k_j}(z^{1,j}, l)} f(t^1, .)\, d\mu(t^1)\right)(y^2)$$

may be different from zero only in the case $B - 1 = k_j$. That is, we have only one j such that $\mathbf{H}_{1,1,A,B} g_j^1$ differs from zero.

See the second case, that is, $y^1 \notin 6I_j^1$. This follows $y^1 \notin I_j^1$, that is, $y^1 \notin \bigcup_{l=\alpha_j^1}^{\beta_j^1} I_{k_j}(z^{1,j}, l)$, and consequently

$$E_A^1 g_j^1(y^1, x^2) = M_A \int_{I_j^1 \cap I_A(y^1)} g_j^1(t^1, x^2)\, d\mu(t^1) = 0$$

for $A > k_j$. In this case the intersection of I_j^1 and $I_A(y^1)$ would be the empty set. Anyhow, $A \leqslant k_j < B$ can be supposed.

By the same method, we have that $\mathbf{F}^1_{1,A,B}g^2_j$ may be different from zero on the complementer set of $6F$ only in the case $B-1\leqslant k_j\leqslant A$. We get this as follows:

$$\begin{aligned}E^1_A g^2_j(y^1,x^2) &= M_A\int_{I^1_j\cap I_A(y^1)}\mu^{-1}(I^2_j)1_{I^2_j}(x^2)\int_{I^2_j} f_j(x^1,t^2)\,d\mu(t^2)\,d\mu(x^1)\\ &=\mu^{-1}(I^2_j)1_{I^2_j}(x^2)M_A\int_{I^1_j\cap I_A(y^1)}\int_{I^2_j} f_j(x^1,t^2)\,d\mu(t^2)\,d\mu(x^1)=0\end{aligned}$$

as if $I^1_j\subset I_A(y)$ (since the integral of f_j is zero), and also in the case when these two intervals are disjoint. That is, $I^1_j\supset I_A(y)$ must be fulfilled (if the integral is not zero), and consequently $k_j\leqslant A$. Suppose this. On the other hand, in this case, if $y^1\notin I^1_j$, then $I^1_j\cap I_A(y)=\emptyset$ after all. So, $y^1\in I^1_j$ is also can be supposed. Therefore, $y^2\notin I^2_j$ (recall that $y\notin 6F$). So, what can be said on $L_{1,B}g^2_j(x^1,y^2)$? If $k_j\leqslant B-2$, then

$$\begin{aligned}L_{1,B}g^2_j(x^1,y^2) &= M_{B-1}\int_{I_{B-1}(y^2)\setminus I_B(y^2)}\mu^{-1}(I^2_j)1_{I^2_j}(x^2)\int_{I^2_j} f_j(x^1,t^2)\,d\mu(t^2)\\ &\quad\times\frac{1}{1-r_{B-1}(y^2-x^2)}\,d\mu(x^2)=0\end{aligned}$$

because $k_j\leqslant B-2$, and $y^2\notin I^2_j$ give that the intersection of the intervals I^2_j and $I_B(y^2_0,\dots,y^2_{B-2},x^2_{B-1})$ is empty for all $x^2_{B-1}\in Z_{\tilde m_{B-1}}$. That is, the only case we are interested (when $\mathbf{F}^1_{1,A,B}g^2_j$ may be different from zero on the complementer set of $6F$) is $B-1\leqslant k_j\leqslant A$.

Now, we follow the proof of Lemma 2.9, and apply inequality (3).

$$\begin{aligned}&\mu\left(\overline{6F}\cap\left(\mathbf{F}^1_1\left(\sum_{j=1}^{\infty}(g^1_j+g^2_j)>\lambda\right)\right)\right)\\ &\leqslant\sum_{A=1}^{\infty}\sum_{B=A-\alpha}^{A+\alpha}\mu\left(\overline{6F}\cap\left((E^1_A\otimes L_{1,B})\left(\sum_{j=1}^{\infty}(g^1_j+g^2_j)>\lambda\right)\right)\right)\\ &\leqslant\sum_{A=1}^{\infty}\sum_{B=A-\alpha}^{A+\alpha}\mu\left(\overline{6F}\cap\left((E^1_A\otimes L_{1,B})\right.\right.\\ &\qquad\times\left.\left.\left(\sum_{\{j:\min\{A,B\}-1\leqslant k_j\leqslant\max\{A,B\}\}}(g^1_j+g^2_j)>\lambda\right)\right)\right)\\ &\leqslant\frac{C}{\lambda}\alpha\sum_{A=1}^{\infty}\sum_{\{j:A-\alpha-1\leqslant k_j<A+\alpha\}}(\|g^1_j\|_1+\|g^2_j\|_1)\\ &\leqslant\frac{C}{\lambda}\alpha^2\|f\|_1.\end{aligned}$$

This inequality, (2), the facts that the two-dimensional operator $\mathbf{F}_1^1$ is of type (L^2, L^2), and $\mu(6F) \leqslant C\|f\|_1/\lambda$ (see Lemma 2.8) by the (already) standard argument give

$$\mu(\mathbf{F}_1^1 f > \lambda) \leqslant \frac{C}{\lambda}(\alpha^2\|f\|_1 + \|f\|_{L\log^+ L}) + \frac{C}{\lambda^2}\|f\|_1.$$

This ends the proof of Lemma 2.10. □

For any $1 \leqslant j \in \mathbb{N}$ define the one-dimensional operator H_j in the following way

$$\begin{aligned} H_j f(y) &:= \sup_{j \leqslant A \in \mathbb{N}} |L_{j,A} f(y)| \\ &:= \sup_{j \leqslant A \in \mathbb{N}} \left| M_{A-j} \int_{\bigcup_{x_{A-j} \neq y_{A-j}} I_A(y_0, \dots, y_{A-j-1}, x_{A-j}, \dots, y_{A-1})} f(x) \right. \\ &\qquad \left. \times \frac{1}{1 - r_{A-j}(y-x)} dx \right|, \end{aligned}$$

where $y \in G_m$. In [13] the author proved the existence of an absolute constant $C > 0$ such that for all $1 \leqslant j \in \mathbb{N}$, $f \in L^1(G_m)$, and $\lambda > 0$

$$\mu(H_j f > \lambda) \leqslant C \frac{j^2\|f\|_1}{2^j \lambda}.$$

Hereinafter, we discuss the two-dimensional version of this operator and inequality. Let

$$\mathbf{H}_{j_1, j_2} f(y^1, y^2) := \sup_{\substack{j_i \leqslant A_i \in \mathbb{N}, i=1,2, \\ |A_1 - A_2| < \alpha}} |(L_{j_1, A_1} \otimes L_{j_2, A_2}) f(y^1, y^2)|$$

for any $f \in L^1(G_m \times G_{\tilde{m}})$, $j_1, j_2 \in \mathbb{P}$ and $y = (y^1, y^2) \in G_m \times G_{\tilde{m}}$. The operator $\mathbf{H}_{j_1, j_2}$ is the generalization of the operator $\mathbf{H}_{1,1}$ we discussed in Lemma 2.6 and in Lemma 2.9. We prove

Lemma 2.11. *For all integrable function $f \in L\log^+ L(G_m \times G_{\tilde{m}})$ we have*

$$\mu(\mathbf{H}_{j_1, j_2} f > \lambda) \leqslant \frac{j_1^3 j_2^3}{2^{j_1+j_2}} \left(\frac{C}{\lambda} \alpha^2 (\|f\|_1 + \|f\|_{L\log^+ L}) + \frac{C}{\lambda^2} \alpha^2 \|f\|_1 \right) \tag{4}$$

for any $\lambda > 0$, $j_1, j_2 \in \mathbb{P}$.

Proof. Basically, the proof is a kind of modification of the one-dimensional case made up by the author [13]. For $j_1 = 1, j_2 = 1$ the proof is nothing else but Lemma 2.9. The proof applies this lemma for a modified two-dimensional Vilenkin group. We apply a finite permutation for the coordinate groups of the Vilenkin group G_m and $G_{\tilde{m}}$ such that for all $A_i \geqslant j_i$, $A_i \in \mathbb{N}$ the $A_i - j_i$th coordinate group and the $A_i - 1$st coordinate group will be adjacent $(i = 1, 2)$. Then we use Lemma 2.9 for the modified group.

$$\mathbf{H}_{j_1,j_2} f(y^1,y^2) \leqslant \sum_{k_1=0}^{j_1-1} \sum_{k_2=0}^{j_2-1} \sup_{\substack{j_i \leqslant A_i \in \mathbb{N} \\ A_i \equiv k_i \bmod j_i \\ i=1,2, |A_1-A_2|<\alpha}} |(L_{j_1,A_1} \otimes L_{j_2,A_2}) f(y^1,y^2)|$$

$$=: \sum_{k_1=0}^{j_1-1} \sum_{k_2=0}^{j_2-1} \mathbf{H}_{j_1,j_2}^{k_1,k_2} f(y^1,y^2).$$

We prove the existence of an absolute constant $C>0$ such that for all $\lambda>0, f \in L^1(G_m \times G_{\tilde{m}})$, and j_i, k_i $(i=1,2)$ the inequality

$$\mu(\mathbf{H}_{j_1,j_2}^{k_1,k_2} f > \lambda) \leqslant \frac{1}{2^{j_1+j_2}} \left(\frac{C}{\lambda} \alpha^2 (\|f\|_1 + \|f\|_{L \log^+ L}) + \frac{C}{\lambda^2} \alpha^2 \|f\|_1 \right) \tag{5}$$

holds. This inequality immediately gives

$$\begin{aligned} &\mu(\mathbf{H}_{j_1,j_2} f > \lambda) \\ &\leqslant \mu\left(\bigcup_{k_1=0}^{j_1-1} \bigcup_{k_2=0}^{j_2-1} \left\{ \mathbf{H}_{j_1,j_2}^{k_1,k_2} f > \frac{\lambda}{j_1 j_2} \right\} \right) \\ &\leqslant \sum_{k_1=0}^{j_1-1} \sum_{k_2=0}^{j_2-1} \mu\left(\mathbf{H}_{j_1,j_2}^{k_1,k_2} f > \frac{\lambda}{j_1 j_2} \right) \\ &\leqslant \frac{j_1^3 j_2^3}{2^{j_1+j_2}} \left(\frac{C}{\lambda} \alpha^2 (\|f\|_1 + \|f\|_{L \log^+ L}) + \frac{C}{\lambda^2} \alpha^2 \|f\|_1 \right) \end{aligned}$$

and this completes the proof of the lemma. Let

$$\mathbf{H}_{j_1,j_2}^{k_1,k_2,N} f(y^1,y^2) := \sup_{\substack{j_i \leqslant A_i \leqslant N j_i + k_i \\ A_i \equiv k_i \bmod j_i \\ i=1,2, |A_1-A_2|<\alpha}} |(L_{j_1,A_1} \otimes L_{j_2,A_2}) f(y^1,y^2)|.$$

Since $\mathbf{H}_{j_1,j_2}^{k_1,k_2,N} f$ is monotone increasing as N gets larger, then by measure theory if we prove that the operators $\mathbf{H}_{j_1,j_2}^{k_1,k_2,N} f$ are of weak type (5), uniformly in N (it means that the constant C does not depend on N, j, k), then $\mathbf{H}_{j_1,j_2}^{k_1,k_2}$ is also of weak type (5). This would complete the proof of the lemma. That is, we have to prove inequality (5) for operators $\mathbf{H}_{j_1,j_2}^{k_1,k_2,N}$.

Recall that the Vilenkin group G_m is the complete direct product of its coordinate groups Z_{m_l}, that is, $G_m = \times_{l=0}^{\infty} Z_{m_l}$, and $G_{\tilde{m}} = \times_{l=0}^{\infty} Z_{\tilde{m}_l}$. We define another pair of Vilenkin groups. Their coordinate groups will be the same, but with certain rearrangement. Let the function $\theta_i : \mathbb{N} \to \mathbb{N}$ be defined in the following way. If $n \geqslant k_i + N j_i$, or $n \not\equiv k_i, k_i - 1 \bmod j_i$, then

$$\theta_i(n) := n$$

and

$$\theta_i(k_i + l j_i) := k_i + (l+1) j_i - 1, \quad \theta_i(k_i + (l+1) j_i - 1) := k_i + l j_i$$

for all $l<N, l\in\mathbb{N}$, and for $i=1,2$. Then define the Vilenkin groups $G_m^{j_1,k_1}$, $G_{\tilde{m}}^{j_2,k_2}$ as

$$G_m^{j_1,k_1}=\mathop{\times}_{l=0}^{\infty} Z_{m_{\theta_1(l)}},\quad G_{\tilde{m}}^{j_2,k_2}=\mathop{\times}_{l=0}^{\infty} Z_{\tilde{m}_{\theta_2(l)}}.$$

We give a measure preserving bijection between the two pairs of Vilenkin groups. We denote it by $\vartheta=(\vartheta_1,\vartheta_2)$, or more precisely (if it is needed) by $\vartheta_{j,k}=(\vartheta_{1,j_1,k_1},\vartheta_{2,j_2,k_2})$. It will not cause any confusion. That is,

$$\vartheta=\vartheta_{j,k}: G_m\times G_{\tilde{m}}\to G_m^{j_1,k_1}\times G_{\tilde{m}}^{j_2,k_2}$$

and let the nth coordinate of the sequence $\vartheta_1(x^1)$ be $x^1_{\theta_1(n)}$, the nth coordinate of the sequence $\vartheta_2(x^2)$ be $x^2_{\theta_2(n)}$. That is, $(\vartheta_{1,j_1,k_1}(x^1),\vartheta_{2,j_2,k_2}(x^2))\in G_m^{j_1,k_1}\times G_{\tilde{m}}^{j_2,k_2}$ Briefly,

$$(\vartheta(x^i))_n=x^i_{\theta(n)}\quad (n\in\mathbb{N},\ i=1,2).$$

Consequently, we have a finite permutation of the coordinates. This is very important for us, since when we discuss the operator $\mathbf{H}_{1,1}$ on the two-dimensional Vilenkin group $G_m^{j_1,k_1}\times G_{\tilde{m}}^{j_2,k_2}$, then we can apply the result given ($\mathbf{H}_{1,1}$ is of weak type (4)) for the operator $\mathbf{H}_{j_1,j_2}^{k_1,k_2,N}$ on the Vilenkin group $G_m\times G_{\tilde{m}}$.

Denote by m^{θ_1} the sequence for which $m_l^{\theta_1}=m_{\theta_1(l)}$, and by m^{θ_2} the sequence for which $m_l^{\theta_2}=\tilde{m}_{\theta_1(l)}$. Introduce the notation $r_l^{\theta_i}:=r_{\theta_i(l)}\,(l\in\mathbb{N}, i=1,2)$. Recall that $A_i\equiv k_i \bmod j_i\ (i=1,2)$. Then we have

$$\begin{aligned}1-r_{A_1-j_1}(y^1-x^1)&=1-\exp\left(2\pi\imath\frac{y^1_{A_1-j_1}-x^1_{A_1-j_1}}{m_{A_1-j_1}}\right)\\&=1-\exp\left(2\pi\imath\frac{\vartheta_1(y^1)_{A_1-1}-\vartheta_1(x^1)_{A_1-1}}{m^{\theta_1}_{A_1-1}}\right)\\&=1-r^{\theta_1}_{A_1-1}(\vartheta_1(y^1)-\vartheta_1(x^1)).\end{aligned}$$

Similarly, $1-r_{A_2-j_2}(y^2-x^2)=1-r^{\theta_2}_{A_2-1}(\vartheta_2(y^2)-\vartheta_2(x^2))$.

Moreover, denote by M^{θ_1}, M^{θ_2} the sequence of the generalized powers with respect to the sequence m^{θ_1} and m^{θ_2}. This gives

$$\begin{aligned}M^{\theta_1}_{A_1-1}&=m_0^{\theta_1}\dots m^{\theta_1}_{A_1-2}\\&=m_0m_1\dots m_{A_1-j_1-1}m_{A_1-j_1+1}\dots m_{A_1-1}\\&=\frac{m_0\dots m_{A_1-1}}{m_{A_1-j_1}}\\&=M_{A_1-j_1}\frac{m_{A_1-j_1}m_{A_1-j_1+1}\dots m_{A_1-1}}{m_{A_1-j_1}}\\&=M_{A_1-j_1}m_{A_1-j_1+1}\dots m_{A_1-1}.\end{aligned}$$

This gives $M_{A_1-j_1} \leqslant M^{\theta_1}_{A_1-1}/2^{j_1-1}$, and similarly $\tilde{M}_{A_2-j_2} \leqslant M^{\theta_2}_{A_2-1}/2^{j_2-1}$. By the above written we get

$$\Bigg| M_{A_1-j_1} \tilde{M}_{A_2-j_2} \int_{\bigcup_{x^1_{A_1-j_1} \neq y^1_{A_1-j_1}} I_{A_1}(y^1_0,\dots,y^1_{A_1-j_1-1},x^1_{A_1-j_1},\dots,y^1_{A_1-1})}$$
$$\times \int_{\bigcup_{x^2_{A_2-j_2} \neq y^2_{A_2-j_2}} I_{A_2}(y^2_0,\dots,y^2_{A_2-j_2-1},x^2_{A_2-j_2},\dots,y^2_{A_2-1})}$$
$$\times f(x^1,x^2) \frac{1}{1-r_{A_1-j_1}(y^1-x^1)} \frac{1}{1-r_{A_2-j_2}(y^2-x^2)} d\mu(x^2)\, d\mu(x^1) \Bigg|$$
$$\leqslant \frac{1}{2^{j_1-1}} \frac{1}{2^{j_2-1}} \mathbf{H}_{1,1} f^\theta(\vartheta_1(y^1), \vartheta_2(y^2)),$$

where the function f^θ is defined on $G^{j_1,k_1}_m \times G^{j_2,k_2}_m$ by $f(x^1,x^2) = f^\theta(\vartheta_1(x^1), \vartheta_2(x^2))$ for all $x \in G_m \times G_{\tilde{m}}$. The definition of $\mathbf{H}^{k_1,k_2,N}_{j_1,j_2} f$ gives

$$\mathbf{H}^{k_1,k_2,N}_{j_1,j_2} f(y^1,y^2) \leqslant \frac{1}{2^{j_1-1}} \frac{1}{2^{j_2-1}} \mathbf{H}_{1,1} f^\theta(\vartheta_1(y^1), \vartheta_2(y^2)).$$

Consequently, by Lemma 2.9 we have

$$\mu(y \in G_m \times G_{\tilde{m}} : \mathbf{H}^{k_1,k_2,N}_{j_1,j_2} f(y) > \lambda)$$
$$\leqslant \mu(\vartheta(y) \in G^{j_1,k_1}_m \times G^{j_2,k_2}_m : \mathbf{H}_{1,1} f^\theta(\vartheta(y)) > \lambda 2^{j_1+j_2-2})$$
$$\leqslant \left(\frac{C}{2^{j_1+j_2-2}\lambda} \alpha^2 (\|f^\theta\|_1 + \|f^\theta\|_{L\log^+ L}) + \frac{C}{(2^{j_1+j+2-2}\lambda)^2} \alpha^2 \|f^\theta\|_1 \right)$$
$$\leqslant \frac{1}{2^{j_1+j_2}} \left(\frac{C}{\lambda} \alpha^2 (\|f\|_1 + \|f\|_{L\log^+ L}) + \frac{C}{\lambda^2} \alpha^2 \|f\|_1 \right).$$

That is, the proof of Lemma 2.11 is complete. □

Define the two-dimensional operators $\mathbf{F}^i_j$ $(i = 1,2, j \in \mathbb{N})$ for integrable functions in the following way:

$$\mathbf{F}^1_{j,A,B} f(y^1,y^2) := (E^1_A \otimes L_{j,B}) f(y^1,y^2), \quad \mathbf{F}^2_{j,A,B} f(y^1,y^2) := (L_{j,A} \otimes E^2_B) f(y^1,y^2),$$

$$\mathbf{F}^i_j f := \sup_{A,B,|A-B|<\alpha} |\mathbf{F}^i_{j,A,B} f|,$$

where $i = 1,2,\ j \in \mathbb{N}$. In the same way as we proved Lemma 2.11 by the application of Lemma 2.9, from Lemma 2.10 we get the proof of the following lemma. □

Lemma 2.12. $\mu(\mathbf{F}^i_j f > \lambda) \leqslant \frac{Cj^3}{2^j}(\frac{1}{\lambda}(\alpha^2\|f\|_1 + \|f\|_{L\log^+ L}) + \frac{1}{\lambda^2}\|f\|_1)$.

Finally, by Lemmas 2.11 and 2.12 we prove the main theorem of this paper, that is Theorem 2.1.

Proof of Theorem 2.1. In the paper of the author [13, Lemma 2.6] one can find the following formula for the Fejér kernels. Let $A > t$, $t, A \in \mathbb{N}$, $z \in I_t(0) \backslash I_{t+1}(0)$. Then

$$K_{M_A}(z) = \begin{cases} 0 & \text{if } z - z_t e_t \notin I_A(0), \\ \dfrac{M_t}{1 - r_t(z)} & \text{if } z - z_t e_t \in I_A(0). \end{cases}$$

Since in the one-dimensional case for $z \in I_A(0)$ we have $K_{M_A}(z) = K_{M_A}(0) = \frac{1}{M_A}\sum_{k=0}^{M_A-1} k = \frac{M_A-1}{2}$, then we have for a $f : G_m \to \mathbb{C}$ integrable function

$$\begin{aligned} \sigma_{M_A} f(y) &= \int_{G_m} f(x) K_{M_A}(y-x)\, d\mu(x) \\ &= \int_{I_A(y)} f(x) K_{M_A}(y-x)\, d\mu(x) + \sum_{t=0}^{A-1} \int_{I_t(y)\backslash I_{t+1}(y)} f(x) K_{M_A}(y-x)\, d\mu(x) \\ &= \frac{M_A - 1}{2} \int_{I_A(y)} f(x)\, d\mu(x) \\ &\quad + \sum_{t=0}^{A-1} M_t \int_{\bigcup_{x_t \neq y_t} I_A(y_0,\ldots,y_{t-1},x_t,y_{t+1}\ldots,y_{A-1})} f(x) \frac{1}{1 - r_t(y-x)}\, d\mu(x). \end{aligned}$$

This immediately gives for the two-dimensional operator $\sigma_{M_{A_1},\tilde{M}_{A_2}}$, where $|A_1 - A_2| < \alpha$

$$\begin{aligned} &|\sigma_{M_{A_1},\tilde{M}_{A_2}} f(y^1,y^2)| \\ &\quad \leqslant \sup_{|A_1 - A_2| < \alpha} |(E^1_{A_1} \otimes E^2_{A_2}) f(y^1,y^2)| + \sum_{i=1}^{2} \sum_{j=1}^{\infty} \mathbf{F}^i_j f(y^1,y^2) \\ &\qquad + \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \mathbf{H}_{j_1,j_2} f(y^1,y^2) \\ &\quad =: f^*_\alpha + \sum_{i=1}^{2} \sum_{j=1}^{\infty} \mathbf{F}^i_j f(y^1,y^2) + \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \mathbf{H}_{j_1,j_2} f(y^1,y^2). \end{aligned}$$

Define the maximal operator

$$\sigma^*_\alpha f := \sup_{\substack{A_1,A_2 \in \mathbb{N} \\ |A_1 - A_2| < \alpha}} |\sigma_{M_{A_1},\tilde{M}_{A_2}} f|.$$

So, by Lemmas 2.12 and 2.11 we have

$$\begin{aligned}
&\mu(\sigma_\alpha^* f > \lambda) \\
&\leqslant \mu(f_\alpha^* > \lambda/3) + \mu\left(\sum_{i=1}^{2}\sum_{j=1}^{\infty} \mathbf{F}_j^i f > \lambda/3\right) + \mu\left(\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty} \mathbf{H}_{j_1,j_2} f > \lambda/3\right) \\
&\leqslant \mu(f_\alpha^* > \lambda/3) + \sum_{i=1}^{2}\sum_{j=1}^{\infty} \mu\left(\mathbf{F}_j^i f > \frac{1}{4j^2}\frac{\lambda}{3}\right) + \sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty} \mu\left(\mathbf{H}_{j_1,j_2} f > \frac{1}{4j_1^2 j_2^2}\frac{\lambda}{3}\right) \\
&\leqslant \mu(f_\alpha^* > \lambda/3) + \sum_{i=1}^{2}\sum_{j=1}^{\infty} \frac{Cj^3}{2^j}\left(\frac{j^2}{\lambda}(\alpha^2\|f\|_1 + \|f\|_{L\log^+ L}) + \frac{j^4}{\lambda^2}\|f\|_1\right) \\
&\quad + \sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty} \frac{j_1^3 j_2^3}{2^{j_1+j_2}}\left(\frac{Cj_1^2 j_2^2}{\lambda}\alpha^2(\|f\|_1 + \|f\|_{L\log^+ L}) + \frac{Cj_1^4 j_2^4}{\lambda^2}\alpha^2\|f\|_1\right) \\
&\leqslant \mu(f_\alpha^* > \lambda/3) + C\alpha^2\frac{1}{\lambda}(\|f\|_1 + \|f\|_{L\log^+ L}) + C\alpha^2\frac{1}{\lambda^2}\|f\|_1.
\end{aligned}$$

It is well known that the unconditional maximal operator $f^* := \sup_{A_1,A_2\in\mathbb{N}}|(E_{A_1}^1 \otimes E_{A_2}^2)f|$ is of weak type $(L\log^+ L, 1)$, that is, $\mu(f^* > \lambda) < C(\|f\|_1 + \|f\|_{L\log^+ L})/\lambda$. One can find a proof for instance in [29]. Hence

$$\mu(\sigma_\alpha^* f > \lambda) \leqslant C\alpha^2\left(\frac{1}{\lambda}(\|f\|_1 + \|f\|_{L\log^+ L}) + \frac{1}{\lambda^2}\|f\|_1\right).$$

Finally, let the integrable function f belong to $L\log^+ L(G_m \times G_{\tilde{m}})$, and $\varepsilon > 0$. Since the set of two-dimensional Vilenkin polynomials is dense in this space we get by standard argument that

$$\mu\left(\limsup_{\substack{A_1,A_2\to\infty \\ |A_1-A_2|<\alpha}} |\sigma_{M_{A_1},\tilde{M}_{A_2}} f - f| > \varepsilon\right) = 0,$$

i.e. that $\limsup_{\substack{A_1,A_2\to\infty \\ |A_1-A_2|<\alpha}} |\sigma_{M_{A_1},\tilde{M}_{A_2}} f - f| = 0$ a.e. The proof of Theorem 2.1 is complete. $\square$

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